REAL ANALYSIS HOMEWORK 1

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1. Problem 1

a. We say that $f_n \to f$ pointwise if for all $x \in S$ and for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ whenever $n > N$.

b. We say that $f_n \to f$ uniformly if for all $\epsilon > 0$ there exists N such that $|f_n(x) - f(x)| < \epsilon$ whenever $n > N$, independent of $x \in S$.

c. Consider the sequence of function $f_n : [0, 1) \to [0, 1)$ such that $f_n(x) = x^n$. It is clear that $f_n \to 0$ pointwise, since $\lim_{n\to\infty} x^n \to 0$ whenever $0 \leq x < 1$ [1].

To show that this convergence is not uniform, set $\epsilon = 1/2$. We wish to show that for any $N \in \mathbb{N}$, there exists $x \in [0, 1)$ such that $x^n \geq \epsilon$. Consider the point $1 - \epsilon/N$. This clearly belongs to $[0, 1)$, and employing Bernoulli's inequality, we see:

$$(1 - \epsilon/N)^N \geq 1 - \epsilon = 1/2$$

For all $N \in \mathbb{N}$, so we are done. [2]

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[1] Quick proof: x^n is decreasing and bounded below by 0, hence tends to 0

[2] For clarification, this is using the negation of part (b): For all $N \in \mathbb{N}$ there exists $\epsilon > 0$ and $x \in S$ such that $|f_n(x) - f(x)| \geq \epsilon$ for all $n \geq N$.

2. Problem 2

a. We say f is continuous at $x_0 \in I$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

f is discontinuous at $x_0 \in I$ if it is not continuous at $x_0 \in I$, ie for all $\delta > 0$ there exists $\epsilon > 0$ such that $|f(x) - f(x_0)| \geq \epsilon$ whenever $|x - x_0| < \delta$.

b. Suppose first that $x = p/q$ is a nonzero rational number. Then, $R(x) = 1/q$, and by density of irrationals we can find an irrational number y for any $\delta > 0$ such that $|x - y| < \delta$. However, if we choose $\epsilon < 1/q$, we see that $|x - y| < \delta$ but $|R(x) - R(y)| = 1/q > \epsilon$, so R is discontinuous at all rational points.

Suppose now that x is irrational. Since, excluding $x = 0$, R is periodic with period 1 (seen by writing $x = \lfloor x \rfloor + \{x\}$), we can assume that $x \in (0, 1)$ without loss of generality. Let $\epsilon > 0$ and find $Q \in \mathbb{N}$ such that $\frac{1}{Q} < \epsilon$. Then, the set of positive integers less than Q is finite, so that we can enumerate all rational numbers in $(0, 1)$ with denominator less than Q as a finite set. [3]

Enumerate this set of rationals $\{p_1/q_1, \dots, p_n/q_n\}$ and merely choose $\delta < \min_i\{|p_i/q_i - x|\}$.

Then, assume $|x - y| < \delta$. If y is irrational, then trivially $|R(x) - R(y)| = 0 < \epsilon$. If $p/q = y \in \mathbb{Q}$, then by construction $|R(x) - R(y)| = 1/q < \epsilon$, so that R is continuous at all rational points.

Finally, if $x = 0$, the exact same argument for irrational x works here as well. Thus, $R(x)$ is continuous at $x = 0$ as well.

[3] The explicit way to do this is to choose $q < Q$ and then construct the set $\{i/q : (i, q) = 1\}$. This will have $\phi(q)$ elements, and we can continue this construction for $q = 1, \dots, Q - 1$.

3. Problem 3

Define $\lim_{y\to b} g(y) := L$. Then, it suffices to show that $\lim_{x\to a} h(x) = L$ in order to complete the problem. Let $\epsilon > 0$. Then, choose δ_i, $i = 1, 2$ such that:

$$|f(x,y) - h(x)| < \epsilon/3$$
$$|f(x,y) - g(y)| < \epsilon/3$$
$$|g(y) - L| < \epsilon/3$$

When $|x-a| < \delta_1$ and $|y-b| < \delta_2$. Then, for $x \in \mathbb{R}$, we have:

$$\begin{aligned} |h(x) - L| &= |h(x) - f(x,y) + f(x,y) - g(y) + g(y) - L| \\ &\leq |f(x,y) - h(x)| + |f(x,y) - g(y)| + |g(y) - L| \qquad (3.1) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Also, the above inequality holds independent of y since we have that $f(x,y) \to h(x)$ uniformly, so that $h(x) \to L$ as $x \to a$, as desired.

4. Problem 4

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Define:

$$U(f,P) := \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$
$$L(f,P) := \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Where $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. From here, define:

$$\overline{\int_a^b} f(x)dx = \inf_P U(f,p)$$

$$\underline{\int_a^b} f(x)dx = \sup_P L(f,P)$$

The upper and lower integrals, respectively. Then, finally, we say that f is Riemann integrable on $[a,b]$ if the upper and lower integrals exist and are equal.

Equivalently, given any $\epsilon > 0$ there exists a partition P such that $U(f,P_0) - L(f,P_0) < \epsilon$ whenever $P \subset P_0$.

5. Problem 5

Let $\epsilon > 0$. Find $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for $n \geq N$. Then, letting $n \geq N$,

$$\begin{aligned} \Big| \int_a^b f_n(x)dx - \int_a^b f(x)dx \Big| &= \Big| \int_a^b (f_n(x) - f(x))dx \Big| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &< \frac{\epsilon}{b-a}(b-a) = \epsilon \end{aligned} \tag{5.1}$$

So that $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.